We are pleased to invite you to attend a three-day course on

Bayesian Assimilation, Updating, and an Introduction to Model Order Reduction Techniques

Dates: September 29 – October 1, 2025
Lecturers: Hermann Matthies (TU Braunschweig) and Giovanni Stabile (Sant'Anna School of Advanced Studies)
Location: Sant'Anna School of Advanced Studies, Pisa
Format: Hybrid (in-person and remote participation possible)
Cost: Free of charge

The course will provide an introduction to Bayesian assimilation and updating methods together with model order reduction techniques, combining theoretical insights with applications.

The event is open to **Master students, PhD candidates, and Postdoctoral researchers** interested in computational science, applied mathematics, engineering, and related fields.

Further details about the program can be found in the attached PDF.

Registration: Attendance is free, but registration is required. Please register by sending an email to:
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Course on Bayesian Assimilation, Updating and an introduction to model order reduction techniques

The course is going to take the example of a diagnostic or predictive digital twin as an example. In such a system there may be quantities of interest (QoIs) which are not directly observable and have to be inferred from the measurement of other, related, quantities and the mathematical resp. computational model of the digital twin. These QoIs may include properties of the digital twin itself, i.e. parameters in the mathematical model. This process is sometimes (lingo from the weather forecast community) called data assimilation. It is usually what is mathematically termed an inverse problem, and it is usually ill-posed. Thus to approach it numerically, it has to be transformed into a well-posed problem. One technique to do this is to use a probabilistic description of the whole procedure and engage in uncertainty quantification (UQ). This goes along with using a probabilistic graphical model to describe the temporal evolution of the physical and digital twin, where the Bayesian directed graph describes the transfer of information in terms of conditional probabilities resp. conditioned random variables. All the parameters appearing in such a mathematical model of the digital twin make this a parametric mathematical model; some of the parameters will describe the probabilistic part used in the Bayesian assimilation. One common theme throughout the course will be the effort to represent possibly complicated mathematical objects in terms of vector spaces, algebras, and linear mappings. One of the reasons for this is that upon concrete representation and discretisation, this becomes a problem in numerical multilinear algebra, which makes it computationally accessible. In the last part of the course we will focus on common techniques to solve parametric problems governed by partial differential equations efficiently. We will focus on both intrusive and non-intrusive approaches.

Time Plan of the Individual Lectures:

	Mon 29/09	**Tue 30/09**	**Wed 01/10**
9:00 - 9:45	Lecture 1	Lecture 4	Lecture 7
9:45 - 10:00	Break	Break	Break
10:00 - 10:45	Lecture 1	Lecture 4	Lecture 7
10:45 - 11:15	Break	Break	Break
11:15 - 12:00	Lecture 2	Lecture 5	Lecture 8
12:00 - 12:15	Break	Break	Break
12:15 - 13:00	Lecture 2	Lecture 5	Lecture 8
12:00 - 14:30	Lunch Break	Lunch Break	Lunch Break
14:30 - 15:15	Lecture 3	Lecture 6	Lecture 9
15:15 - 15:30	Break	Break	Break
15:30 - 16:15	Lecture 3	Lecture 6	Lecture 9

- Room 4, Palazzo Boyl, Via Santa Cecilia, 24, Pisa
- Room 6, Sant'Anna Main Campus, Piazza Martiri della Libertà, 33, Pisa

A description of the individual lectures follows:

Lecture 1: Analysis of Parametric Models - Lecturer: Prof. Hermann G. Matthies - TU Braunschweig

Parametric models are understood to be functions of some parameters with values in a vector space. Examples are parameter dependent operators — linear and non-linear, parametric right-hand sides or forcing terms, boundary conditions, parametric matrices, and, last but not least, the solutions to parametric equations. The parameters in question could be used for e.g. optimisation, control, describing different environments, or they could be uncertain. Abstractly, such a map from a set of parameters into a vector space can be associated with a linear map, which carries the same information and is in some cases a true generalisation. A linear map is a much more structured object than a general map from aparameter space into a vector space, and allows one to use advanced analytical tools for their analysis, such as various decompositions and expansions, like the SVD (singular value decomposition) and spectral analysis. It also shows a direct link to reproducing kernel Hilbert spaces (RKHS), as the set of parameters is replaced by a vector space of real valued functions, again an object with structure. These techniques lead to tensor representations, which effectively introduce a new set of parameters — these real valued functions — so that the parametric map is linear resp. affine in these new parameters. In many reduced order methods (ROMs), e.g. the reduced basis technique, this is an important pre-condition, which can be achieved with the proposed analytical tools for any parametric object. Allowing for a bit more structure on the set of parameters, one can introduce analogues of the correlation operator and its kernel twin. Factorisations of these operators provides additional representations, and in fact any factorisation corresponds to a representation of the parametric map. The well known Karhunen-Loève expansion is one example of this. Allowing for a bit more structure on the side of the vector space everything is mapped to, the focus changes on how to preserve that kind of structure in a tensor-product like approximation. We will touch on symmetries, positivity (positive definite matrices), and Lie groups like the orthogonal group.

Lecture 2: Algebras and Vector Spaces of Random Variables - Lecturer: Prof. Hermann G. Matthies - TU Braunschweig

Assuming that the participants know the basics of probability theory, this will be a refresher stressing an algebraic view of probability — such as it is also used e.g. in quantum computing. But at the start there will be short discussion of uncertainty in general, and different proposals — touching a bit on the history of the subject — on how to model it mathematically, and the properties and possibilities of such different models. Hilbert, in his 1900 so-called "problems lecture", formulated as 6th problem the challenge to find an axiomatic basis for mechanics and probability. Kolmogorov's 1933 "Grundbegriffe" monograph was widely accepted as an adequate answer to this challenge regarding the axiomatisation of — one has to say now — "classical" probability. Coincidentally, 1900 is also the year when Planck formulated his thesis of energy quanta, which would give rise to quantum theory, and which requires a new probability theory. This became clear after Heisenberg's 1925 paper and his Göttingen colleagues' works afterwards, which together with Dirac introduced the algebraic point of view, culminating in von Neumann's 1932 monograph on the widely known Hilbert space representation of quantum mechanics; which

actually appeared before Kolmogorov's monograph. The main difference between the classical and the "new" probability lay in the non-commutativity of random variables. Today there are also other areas known where such quantum like behaviour (QLB) seems to occur. The algebraic view offers a way on how to treat both classical andquantum like phenomena in a unified mathematical setting. And although probabilists today seem to be happy with Kolmogorov's approach based on measure theory, it may be interesting to look at the subject through a different pair of glasses. This algebraic view also offers a more direct way to address random variables with values in infinite dimensional spaces, something which with classical measure theory can only be done in a somewhat circumlocutory fashion. It also helps to separate purely algebraic questions from analytical ones, but of course thrives in the interplay of both. Without wanting to present a strict axiomatic derivation, the start will be an early — and in the light of modern theory also abstract algebraic — view on random variables, as can be found implicitly in the work of early probabilists like the Bernoullis. Their properties are sketched as emanating from simple operational requirements regarding random variables, the mean or expectation (also called the “state” in physics lingo), as well as sampling or observations. Concrete representations of this abstract setting connect it with algebras of linear mappings and the spectral theory of these, and one may recover Kolmogorov's classical characterisation as one particular representation. The basic foundation will thus not be the Kolmogorovean model of a measure space and a probability measure, but rather algebras of random variables (RVs) together with the expectation functional. This will be used to bootstrap to a basic Hilbert space of RVs, and associated Banach spaces. This introduces a functional analytic setting and topologies which give rise to notions of convergence, and new RVs as limits in the completion of basic RV algebras. The basic descriptors — mathematical representations — of random variables will be discussed. Well-known notions like correlation and independence will be given an algebraic resp. geometric interpretation. One important subject is how to compute functions of random variables beyond polynomials, something which is inherent in the definition of an algebra. Here the representation of RVs as liner mappings combined with spectral calculus will be employed. The representation of RVs as linear mappings also leads to the quantum decomposition of a random variable, using the number operator as well as ladder operators (creation, annihilation, preservation operator).

Lecture 3: Vector Valued Random Variables, Random Fields, and Basic Algorithms for Uncertainty Quantification - Lecturer: Prof. Hermann G. Matthies - TU Braunschweig

The description of vector valued random variables will continue the discussion started for parametrised models. Here the parameters are elements of a probability space in general, and more often specifically other random variables. Special cases of this general construction are random fields, as well as random fields of more structure, e.g. random fields of vectors, tensors, positive definite matrices, orthogonal matrices, and elements of Lie groups in general. Correlation and covariance operators of such objects will be considered, as well as their spectral analysis. This will continue the analysis ofthe first lecture, and also touch on the important subject of symmetries and how to deal with them, if one wants to preserve this under numerical approximation. This then leads to methods of describing random processes and fields, as well as probability distributions on infinite dimensional vector spaces, as well as to representations by higher order tensors which then opens the possibility for low-rank tensor approximations. The basic algorithmic or numerical problem addressed here is the so- called forward problem: given a mathematical problem involving

uncertainty in the form input random variables, how does this affect the result, i.e. the solution of the governing equations. The algorithms addressed will be probabilistic ones, like those of the Monte Carlo family, as well as deterministic ones. Depending on which kind of descriptor one chooses for representation, different formulations and thus algorithms are suggested. This includes perturbation methods, the ones based on the Fokker-Planck equation, and the Monte Carlo (MC) and quasi Monte Carlo (QMC) ones. The deterministic ones will lean heavily on the functional analytic setting introduced in the previous lecture, and will remind many of the methods used to deal with partial differential equations, like Galerkin and collocation methods.

Lecture 4: Inverse Problems and Bayesian Assimilation or Updating - Lecturer: Prof. Hermann G. Matthies - TU Braunschweig

Inverse problems, and the connection to ill- and well-posed problems. Probabilistic approaches are based fundamentally on Bayes's theorem, resp. more modern notions of conditioning. Conditional expectation will be used as the central concept. It will be introduced as projection, or even as a best approximation in a Hilbert space setting, a minimisation of a quadratic functional over a linear subspace — very reminiscent of Galerkin approximations. Connections with the classical theorems of Bayes and Laplace on conditional probability will be sketched, as well as the one with the notion of Radon-Nikodym derivative. The assimilation or updating step will be seen as change of expectation functional and thus the probability measure, or any of the other descriptors of random variables introduced previously. Depending on which kind of descriptor one wants to update, different formulations and thus algorithms are suggested. Historically the first ones were probabilistic ones based on sampling a Markov chain, which lead to Markov chain Monte Carlo algorithms. Completely different possibilities arise from the functional analytic view, and one can design algorithms based on Galerkin approximations. These algorithms change the probability measure to the conditioned one. In the lingo of physics, this can be seen as the Schrödinger picture, as the state changes. In parallel, and initially not connected to Bayesian updating, filtering algorithms were developed, such as the Kalman filter. The filtering algorithms do the update by changing the random variable, and leaving the basicexpectation functional, the state, untouched. This can be viewed as the Heisenberg approach in physics lingo, as the observable changes.

Lecture 5: Bayesian Assimilation and Filtering, Proxy Models as Conditional Expectation - Lecturer: Prof. Hermann G. Matthies - TU Braunschweig

Common filtering algorithms will be described as approximations to the conditional expectation. When focusing on the random variable itself or its functional representation as a function of other, known, RVs, instead of on any of the other descriptors, the idea of adjusting the RV to represent the new information suggests itself. The simplest kind of approximation turns out to be a linear one, and it connects with the Gauss-Markov theorem. This leads to linear filtering, especially to the Kalman filter (KF) and its further developments. Although the Kalman filter was, after connecting it with Bayesian updating, originally seen to only apply to Gaussian RVs, it is possible to extend the idea to the more general situation, resulting in the abstract Gauss-Markov-Kalman filter (GMKF). One by now well known implementation of the GMKF — although originally derived differently — is its Monte Carlo discretisation in form of the Ensemble Kalman filter (EnKF). A discretisation in the functional

analytic setting leads to a numerical realisation on Wiener's polynomial chaos, resulting in the polynomial chaos GMKF. It is also possible to go beyond linear filters, although this is rarely done. In an abstract setting, all these constructions are attempts to invert a non-invertible — usually high dimensional — mapping in some specific way reminiscent of conditional expectation, as described previously. By extending this idea, mainly probabilistic proxy models such as Kriging and Gaussian process emulation (GPE) can be seen as instances of approximate computation of a conditional expectation. The same is true of many of the models proposed for machine learning, such as deep neural networks. Similar remarks apply to the determination of reduced order models (ROMs). This point of view allows one to take other uncertainties, usually not considered in the computation of proxy models, into consideration. It also allows to use other measures of optimality in the computation.

Lecture 6: Topological Vector Spaces of RVs and a View at Non-Commuting Algebras of RVs. - Lecturer: Prof. Hermann G. Matthies - TU Braunschweig

To have a regularity theory for probabilistic equations — mirroring the regularity theory for partial differential equations, which is important in order to estimate possible convergence speeds of numerical discretisations — it is important to introduce notions of regularity and smoothness of RVs. This will be done by reviewing the methods used in the analysis of functions on finite dimensional vector spaces.This will establish connections with the theory of Schwartz distributions and generalised or wild RVs not representable as measurable maps. In the course of this Gelfand triples and chains of spaces of RVs will be used to construct countably Hilbertian spaces and ultimately nuclear spaces. This can be used to construct generalised random variables and "ideal elements". It allows the specification of not only analogues of all the classical spaces of random variables, but to go beyond this and address questions of "smoothness" on the one hand, and the definition of idealised elements resp. "generalised" random variables on the other hand. This very much echoes the construction of distributions resp. generalised functions in the sense of Sobolev and Schwartz. We shall touch on generalised polynomial chaos, the Hida-Kondratiev and Kondratiev spaces, and allude to the connections with Malliavin calculus and Malliavin-Sobolev spaces. This also connects with the iInteracting Fock-space view introduced earlier in the quantum decomposition of RVs using the number operator and associated ladder operators Turning to the subject of non-commutative algebras of RVs, striking differences between classical or commutative probability and non-commutative probability appear already with simple linear algebra. And possible novel devices like quantum computers or quantum information channels can be described in this setting. Such devices may soon be a possibility to use in scientific computing. This part of the lecture will only touch these subjects, but will try to introduce the main concepts on the basis of the theory outlined in the previous lectures.

Lecture 7: Introduction to dimensionality reduction techniques - Lecturer: Prof. Giovanni Stabile - Sant'Anna School of Advanced Studies

This lecture introduces modern strategies for the efficient approximation of parameter-dependent partial differential equations (PDEs). The first part covers the mathematical formulation of parametric PDEs and the need for reduced-order modeling in computational science and engineering. Classical linear reduction methods are then

presented: Proper Orthogonal Decomposition (POD) for extracting low-dimensional structures from data, and greedy algorithms for the adaptive construction of reduced spaces with rigorous error control. Building on these foundations, the course explores nonlinear and data-driven reduction techniques, with a focus on autoencoders as a neural-network-based generalization of POD. Throughout the course, theoretical insights are complemented by algorithmic perspectives and computational examples, enabling students to compare projection-based and machine learning approaches for reduced modeling of complex parametric problems.

Lecture 8: Approximation of the evolution of the latent coordinates - Lecturer: Prof. Giovanni Stabile - Sant'Anna School of Advanced Studies

This lecture addresses non-intrusive strategies for reduced-order modeling of parametric partial differential equations, focusing on the approximation of latent coordinate dynamics obtained from projection- or data-driven model reduction techniques. Rather than relying on intrusive Galerkin projections of the governing equations, the evolution of reduced variables is reconstructed directly from data through regression-based surrogate models. We present and compare several approaches: Radial Basis Function (RBF) interpolation, offering a flexible mesh-free approximation scheme; Gaussian Process Regression, which combines probabilistic modeling with uncertainty quantification; and neural networks, which enable highly expressive, nonlinear mappings. Emphasis is placed on the trade-offs between accuracy, computational efficiency, and generalization capability across the parameter space. Applications to fluid and transport problems illustrate how these non-intrusive methods can extend reduced-order modeling to complex systems where classical intrusive approaches are impractical.

Lecture 9: Introduction to POD Galerkin Methods and the reduced basis method - Lecturer: Prof. Giovanni Stabile - Sant'Anna School of Advanced Studies

This lecture introduces intrusive projection-based techniques for the reduced-order modeling of parametric partial differential equations. We first present the Proper Orthogonal Decomposition (POD)-Galerkin framework, where reduced bases are extracted from solution snapshots and coupled with Galerkin projection to yield low-dimensional dynamical systems. We then discuss the Reduced Basis (RB) methodology, emphasizing its rigorous treatment of parameter dependence, certified error bounds, and offline–online decomposition for efficient many-query contexts. A central challenge in these methods lies in the efficient treatment of nonlinear terms; to this end, the lecture covers the Discrete Empirical Interpolation Method (DEIM) and related hyper-reduction strategies, which enable significant computational savings without compromising accuracy. Throughout, theoretical foundations are complemented by algorithmic insights and illustrative applications, highlighting the strengths and limitations of intrusive approaches compared to emerging non-intrusive alternatives.